FOR WHAT FILTERS IS EVERY REDUCED PRODUCT SATURATED?

BY

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ABSTRACT

In this paper we shall characterize the filters D such that for every M_i, $\Pi_{i\in I} M_i/D$ is λ-saturated (where $\lambda > \aleph_0$). The characterization is: D is λ-good, D is $\aleph_0$ incomplete and $S(I)/D$ is λ-saturated.

0. Introduction

In [4] Keisler finds a pure set-theoretic condition on ultrafilters called λ-goodness such that: for every model M_i, $\Pi M_i/D$ is λ-saturated iff D is λ-good, and ω-incomplete ($\lambda > \aleph_0$). Naturally the question arises: For what filters D is $\Pi_{i\in I}M_i/D$ λ-saturated for every M_i.

Jonsson and Olin [13] give a partial answer by proving that if $I = \omega$ and D is the set of subsets of I with finite complements, then $\Pi_{i\in I} M_i/D$ is $\aleph_1$-saturated. They then asked the above mentioned question. Galvin [11] generalizes their results.

Benda [1] characterized the filters D for which $\Pi_{i\in I} M_i/D$ (for every M_i) satisfies the following condition: Any type on $\Pi_{i\in I}M_i/D$ consisting of quantifier-free formulas (with parameters from $\Pi_{i\in I} M_i/D$) which is finitely satisfiable and of cardinality $<\lambda$ is satisfiable in $\Pi_{i\in I} M_i/D$. (Our proof also gives the answer for the case of Π_n or Σ_n formulas. See the abstract in [12] in which our results were announced.)

Pacholski and Ryll-Nardzewski [9] deal with the case where $S(I)/D$ is atomless for $\aleph_1$-saturation. At about the same time that I proved the theorem appearing here, Pacholski proved a similar theorem for the case $\lambda = \aleph_1$ (see [14]).

We shall use here the following theorem, which is an immediate consequence of theorem 3.1 of Feferman and Vaught [10].

THEOREM 0.1. *For every formula $\phi(x_1, \cdots, x_n)$ in a language L, there exists*

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a formula ψ *(in* L_1*, the language of boolean algebra) and* $\phi_0(x_1,\cdots,x_n),\cdots,\phi_m(x_1,\cdots,x_n)$ *(in L) such that* $\Pi M_i/D \models \phi[\bar{a}]$ *iff* $S(I)/D \models \phi[A_1,\cdots,A_m]$ *where* $A_j = \{i : M_i \models \phi_j[\bar{a}(i)]\}$.

We prove here

THEOREM 0.2. *For every* M_i, $i \in I$, $\Pi_{i \in I} M_i/D$ *is* λ*-saturated where* $\lambda > \aleph_0$ *iff D satisfies the following three conditions*

(1) *D is* ω*-incomplete*

(2) $S(I)/D$ *is* λ*-saturated*

(3) *D is a* λ*-good filter.*

REMARK. We also find when M^I/D is λ-universal for every M.

1. Notations and definitions

λ, μ will denote infinite cardinals; m, n, p natural numbers, and j, k, l ordinals. Let L be a first-order language. Formulas will be denoted by θ, ϕ, ψ, χ. M, N will denote models (L-models if not mentioned otherwise). B will denote a Boolean algebra whose operations are $y_1 \bigcup y_2$, $y_1 \bigcap y_2$, y_1^c. If M is an L-model, then $L(M) = L$ and $L_1 = L(B)$. B_0 is the Boolean algebra with two elements.

An element of a model M will be denoted by a, b, c. The set of elements of M is $|M|$. We shall write many times $a \in M$ instead of $a \in |M|$. $\bar{a}$ will denote a finite sequence of elements of M. $\bar{a} \in M$ if $\bar{a} = \langle a_0,\cdots,a_n\rangle$ and $a_0,\cdots,a_n \in |M|$. $M \models \phi[\bar{a}]$ if M satisfies $\phi[\bar{a}]$.

$S(I)$ will be the set of subsets of I. Clearly with the operations of union, intersection and complementation it is a Boolean algebra.

$D \subset S(I)$ is a filter on I if

1) $A_1, A_2 \in D \Rightarrow A_1 \cap A_2 \in D$

2) $A_1 \in D$, $A_1 \subset A_2 \subset I \Rightarrow A_2 \in D$

D will always denote a filter on I, and i an element of I.

If $A, B \in S(I)$, let $A = B \pmod D$ if $\{i : i \in A \Leftrightarrow i \in B\} \in D$. This is an equivalence relation. So let A/D be the equivalence class of A. Let $A/D \cup A_1/D = (A \cup A_1)/D$, $A/D \cap A_1/D = (A \cap A_1)/D$, $(A/D)^c = (I - A)/D$. It is easily proved that these operations are not dependent on the representatives of the equivalence classes. $B(D)$ will be the Boolean algebra with $\{A/D : A \in S(I)\}$ as a set of elements (the operation has been defined before). (But we shall usually write A instead of A/D.)

The definitions of reduced power and reduced product are assumed to be known (see [2]). If $\bar{a} = \langle a_0,\cdots,a_n\rangle \in M^I/D$, then $\bar{a}(i) = \langle a_0(i),\cdots,a_n(i)\rangle$. $S_\mu(\lambda)$ is the set of all subsets of λ of power $< \mu$. $f: S_\omega(\lambda) \to D$ is monotonic if for every $s, t \in S_\omega(\lambda)$, $s \subset t$ implies $f(t) \subset f(s)$. f is multiplicative if for every $s, t \in S_\omega(\lambda)$, $f(s \cup t) = f(s) \cap f(t)$. $f \subset g$ if for every $s \in S_\omega(\lambda)$, $f(s) \subset g(s)$. D is a μ-good filter if for every monotonic f: $S_\omega(\lambda) \to D$, $\lambda < \mu$, there is g, $g \subset f$, $g: S_\omega(\lambda) \to D$, and g is multiplicative.

D is ω-incomplete if there exists $A_n \in D$ such that $\bigcap_{n<\omega} A_n = 0$.

M is λ-saturated if every type $\{\phi_k(x, \bar{a}_k): k < k_0 < \lambda\}$

($\bar{a}_k \in M$) which is finitely satisfiable in M, is satisfiable in M. M is λ-universal if every elementarily equivalent model of power $\leqq \lambda$ has an isomorphic elementarily submodel of M.

2. On saturative filters

DEFINITION 2.1. D is λ-saturative if for every model M_i, $\Pi_{i \in I} M_i/D$ is λ-saturated.

The main theorem of this paper is:

THEOREM 2.1. D is *λ-saturative* ($\lambda > \aleph_0$) *iff it satisfies the following conditions*:

(1) *D is λ-good*

(2) *$B(D)$ is λ-saturated*

(3) *D is ω-incomplete.*

REMARKS. (1) The proof is separated into several lemmas. (2) In the proof of the necessity, we assume only that for every M, M^I/D is λ-saturated; hence this is equivalent to saturativeness.

PROOF OF NECESSITY

LEMMA 2.2. *If D is λ-saturative and $\lambda > \aleph_0$, then D is ω-incomplete.*

PROOF. Let M be a model $|M| = S_\omega(\omega)$, and let inclusion be one of its relations. The type $\{x \supset \{n\}: n \in \omega\}$ clearly is finitely satisfied in M^I/D, and so it should be realized by a, $a \in M^I/D$. Let $X_n = \{i: a(i) \supset \{n\}\}$. Clearly each X_n belongs to D (by the definition of reduced power). $\bigcap_n X_n = \{i: (\forall n \in \omega) a(i) \supset \{n\}\}$ $= 0$ as no element of M is an infinite set.

LEMMA 2.3. *If D is λ-saturative, then $B(D)$ is λ-saturated.*

PROOF. Let B_0 be the Boolean algebra of two elements. It is easily seen that B_0^I/D is isomorphic to $B(D)$. By hypothesis B_0^I/D is λ-saturated; hence $B(D)$ is λ-saturated.

LEMMA 2.4. *If D is λ-saturative, then it is λ-good.*

REMARK. This proof appears essentially in Keisler [4].

PROOF. Let $\mu < \lambda$, and f: $S_\omega(\mu) \to D$ be monotonic. Then let $M = \langle S(S_\omega(\mu)), \supset, 0, \neq \rangle$. We define

$$a_k,\ k < \mu,\ a_k \in M^I/D, \text{ such that } q = \{a_k \supset x \wedge x \neq 0 : k < \mu\}$$

will be finitely satisfiable, and will be satisfiable iff there exists multiplicative g, $g \subset f$, g: $S_\omega(\mu) \to D$. Clearly this will prove the Lemma.

Let $a_k(i) = \{t : t \in S_\omega(\mu),\ k \in t,\ i \in f(t)\}$. We prove that q is finitely satisfiable. If q_1 is a finite subset of q, then for some s, $q_1 = \{a_k \supset x \wedge x \neq 0 : k \in s\}$. So we should prove that $M^I/D \models (\exists x) \bigwedge_{k \in s}(a_k \supset x \wedge x \neq 0)$ or that

$$A = \{i : M \models (\exists x) \bigwedge_{k \in s} (a_k(i) \supset x \wedge x \neq 0)\} \in D.$$

Clearly,

$$\begin{aligned} A &= \{i : \bigcap_{k \in s} a_k(i) \neq 0\} \\ &= \{i : \{t : t \in S_\omega(\mu),\ k \in s \Rightarrow k \in t,\ i \in f(t)\} \neq 0\} \\ &= \{i : i \in f(s)\} = f(s) \in D. \end{aligned}$$

Hence we have proved that q is finitely satisfiable. As $|q| = \mu < \lambda$ and D is λ-saturative, q is satisfied in M^I/D. Let a realize q, and define $g : S_\omega(\mu) \to D$:

$$g(s) = \{i : M \models \bigwedge_{k \in s} (a_k(i) \supset a(i) \wedge a(i) \neq 0)\}.$$

As $\quad M \models \bigwedge_{k \in s} (a_k(i) \supset a(i) \wedge a(i) \neq 0) \to (\exists x) \bigwedge_{k \in s} (a_k(i) \supset x \wedge x \neq 0),$

clearly $g(s) \subset f(s)$. It is also easily seen that $g(s) \in D$ and that $g(s)$ is multiplicative.

So we have proved the Lemma and hence the necessary part of the proof.

PROOF OF SUFFICIENCY. We assume now that D satisfies the three conditions mentioned in the statement of Theorem 2.1.

LEMMA 2.5. *Let f be a function with two places, whose domain is $S_\omega(\mu)$, $\mu < \lambda$, and whose range is included in $S(I)$. Let $\langle A_i : i < \mu \rangle$ be a sequence of subsets of I. Suppose that*

(1) *for every* $s, t \in S_\omega(\mu), \bigcap_{j \in s} A_j \cap \bigcap_{j \in t}(I - A_j) \subset f(s,t) (\mathrm{mod}\, D)$, *and if* $s \cap t \neq 0, f(s,t) = 0$.

(2) $f(0,0) = I$, *and for every* $s, t, u \in S_\omega(\mu)$, $s \cup t \subset u$, $s \cap t = 0$ *the following holds*:

$$f(s,t) = \cup\{f(s_1,t_1): s \subset s_1 \subset u, t \subset t_1 \subset u, s_1 \cup t_1 = u\}$$

Then there exists a sequence $\langle A_i^1 : i < \mu\rangle$ *of subsets of* I *such that*

(A) *for every* $i < \mu$, $A_i^1 = A_i \pmod D$

(B) *for every* $s, t \in S_\omega(\mu)$

$$\bigcap_{j\in s} A_j^1 \cap \bigcap_{j \in t} (I - A_j^1) \subset f(s,t)$$

REMARK. In this proof we use only the λ-goodness of D.

PROOF. Define for every t, $s \in S_\omega(\mu)$

$$f_1(s,t) = f(s,t) \cup [\bigcap_{j\in s} A_j \cap \bigcap_{j\in t} (I - A_j)]^c.$$

By hypothesis $f_1(s,t) \in D$. For every $u \in S_\omega(\mu)$, let

$$g(u) = \cap\{f_1(s,t): s \cup t \subset u\}.$$

Clearly $g: S_\omega(\mu) \to D$ and g is monotonic. As D is λ-good, $\mu < \lambda$, there exists $h: S_\omega(\mu) \to D$, which is multiplicative and $s \in S_\omega(\mu) \Rightarrow h(s) \subset g(s)$.

In order to satisfy (A) and (B), it is sufficient that the A_i^1 will satisfy:

(α) if $i \in h(\{k\})$ then $i \in A_k^1$ iff $i \in A_k$.

(β) if $i \in A_k^1$ for every $k \in s$, and $i \in I - A_k^1$ for every $k \in t$ then $i \in f(s,t)$.

Let us show the sufficiency. Clearly condition (α) implies (A), i.e., that $A_k = A_k^1 \pmod D$, because $h(\{k\}) \in D$. It is also clear that (β) implies (B).

Now the conditions (α), (β) are local, i.e., they can be solved separately for each i. Let $i_0 \in I$. So it is sufficient to prove that we can define $\langle A_k^1 \cap \{i_0\}: k < \mu\rangle$ such that (α) and (β) are satisfied for $i = i_0$. We can look at $\{i_0 \in A_k^1 : k < \mu\}$ as a set of propositional variables,

$$\{i_0 \in f(s,t): s, t \in S_\omega(\mu)\} \cup \{i_0 \in h(\{k\}): k < \mu\} \cup \{i_0 \in A_k : k < \mu\}$$

as a set of propositional constants, and (α) and (β) as a set of formulas. It is clear that in order to end the proof of the lemma, it is sufficient to prove that this set is consistent. By the compactness theorem for propositional calculus, it is sufficient to prove that this set is finitely satisfiable. Let T_1 be such a finite subset, and u be the set of k for which A_k, or $h(\{k\})$ or A_k^1, or $f(t_1, t_2)$ (where $k \in t_1$ or $k \in t_2$) appear in at least one of the formulas in T_1. Clearly $u \in S_\omega(\mu)$. Let $s = \{k: k \in u,\ i_0 \in h(\{k\}),\ i_0 \in A_k\}$, $t = \{k: k \in u,\ i_0 \in h(\{k\}), i_0 \notin A_k\}$. Then $s \cup t \subset u$, and $s \cap t = 0$, and, as h is multiplicative, $i_0 \in h(s \cup t)$.

It is known that

$$f(s,t) = \cup\{f(s_1,t_1): s \subset s_1 \subset u,\ t \subset t_1 \subset u,\ s_1 \cup t_1 = u\}$$

It is also known that $i_0 \in A_k$ for $k \in s$ and $i_0 \in I - A_k$ for $k \in t$. So $i_0 \in \bigcap_{k \in s} A_k \cap \bigcap_{k \in t}(I - A_k)$ and hence $i_0 \in f(s,t)$. We can conclude that there exist s_1, t_1 $s \subset s_1 \subset u$, $t \subset t_1 \subset u$, $s_1 \cup t_1 = u$ such that $i_0 \in f(s_1,t_1)$. As $f(s_1,t_1) \neq 0$, $s_1 \cap t_1 = 0$.

If we make the formulas in $\{i_0 \in A_k^1 : k \in s_1\}$ true, and make the formulas in $\{i_0 \in A_k^1 : k \in t_1\}$ false, this will make any formula in T_1 true. By this we finish the proof of Lemma 2.5.

LEMMA 2.6. For every $\mu < \lambda$, D is (ω,μ)-regular, i.e., there exist μ sets in D, such that the intersection of any infinite number of them is void.

PROOF. As D is ω-incomplete, there exist $A_n, n < \omega, A_n \in D$, and $\bigcap_{n<\omega} A_n = 0$. Let us define a function $f, f: S_\omega(\mu) \to D$. $f(s) = \cap\{A_n : n \leqq |s|\}$. It is easily seen that f is monotonic, and $f(s) \in D$. Hence there exists $g, g: S_\omega(\mu) \to D$, g multiplicative, and for every s, $g(s) \subset f(s)$. $K = \{g(\{k\}): k < \mu\}$ is a set of μ sets in D. We shall prove that the intersection of any infinite number of sets of K is void, and by this prove the lemma. Otherwise there exists a $k < \mu$ which belongs to $g(\{k_1\}, g(\{k_2\}), \cdots$ where $n \neq m \Rightarrow k_n \neq k_m$. As $\bigcap_{n<\omega} A_n = 0$, there exists m such that $k \notin A_m$, but

$$k \in g(\{k_1\}) \cap g(\{k_2\}) \cap \cdots \cap g(\{k_m\}) = g(\{k_1, \cdots, k_m\}) \subset A_m,$$

a contradiction.

We now proceed to complete the proof of sufficiency.

Let $N = \prod_{i \in I} M_i / D$.

Suppose $q = \{\phi_k(x, \bar{a}_k): k < \mu\} (\mu < \lambda)$ is finitely satisfiable in N (and $\bar{a}_k \in N$). We should prove that q itself is satisfiable. We first try to translate the problem to problems about each i.

Let, for $s \in S_\omega(\mu)$, $\theta_s(x, \bar{a}_s) = \bigwedge_{k \in s} \phi_k(x, \bar{a}_k)$. By Theorem 0.1, for each s there exist $\psi_s(y_s^1, \cdots, y_s^{m_s}), \theta_{\langle s,1\rangle}(x, \bar{a}_{\langle s,1\rangle}), \cdots, \theta_{\langle s,m_s\rangle}(x, \bar{a}_{\langle s,m_s\rangle})$, $(\bar{a}_{\langle s,j\rangle} = \bar{a}_s)$, such that for each $b \in N$

$$N \models \theta_s[b, \bar{a}_s] \text{ iff } B(D) \models \psi_s[I(\theta_{\langle s,1\rangle}[b, \bar{a}_{\langle s,1\rangle}]), \cdots]$$

where $I(\theta(a, \cdots)) = \{i : M_i \models \theta(a(i), \cdots)\}$

Let $R_s = \{\langle s,j\rangle : j = 1, \cdots, m_s\}$, and $R = \cup\{R_s : s \in S_\omega(\mu)\}$. We define for $w, v \in S_\omega(R)$,

$$f(w,v) = I((\exists x)\,[\bigwedge_{h \in w} \theta_h(x,\bar{a}_\epsilon) \wedge \bigwedge_{h \in v} \neg\, \theta_h(x,\bar{a})]).$$

It is easy to see that f satisfies condition (2) from Lemma 2.5 (with R instead of μ).

Now we shall prove that:

(*) *There exist subsets Y_r of I, for $r \in R$, such that:*

(1) *for each $s \in S_\omega(\mu)$, $B(D) \models \psi_s[Y_{\langle s,1\rangle}, Y_{\langle s,2\rangle}, \cdots,]$*

(2) *for each $w,\ v \in S_\omega(R)$, $B(D) \models \bigcap_{r \in w} Y_r \cap \bigcap_{r \in v} Y_r^c \subset f(w,v)$.*

The number of formulas appearing in (1) and (2) is $\leqq |S_\omega(\mu)| + |S_\omega(R) \times S_\omega(R)| = \mu + \mu^2 = \mu < \lambda$. Hence, (since $B(D)$ is λ-saturated), in order to prove (*), it is sufficient to prove that any finite set of formulas appearing in (1) and (2) is satisfiable. Suppose the set is

$$\{\psi_s[Y_{\langle s,1\rangle}, Y_{\langle s,2\rangle}, \cdots,]: s \in S\} \cup \{ \bigcap_{r \in w} Y_r \cap \bigcap_{r \in v} Y_r^c \subset f(w,v): \langle w,v\rangle \in T\}$$

where S and T are finite. Let $t = \{k: k \in s \in S\} \cup \{k: k \in s, \langle s,j\rangle \in w \cup v, \langle w,v\rangle \in T\}$. Clearly t also is finite, and $t \subset \mu$. By our assumption on q, $\{\psi_k(x,\bar{a}_k): k \in t\}$ is satisfiable, say by b. Hence for every $s \subset t$, $N \models \theta_s(b,\bar{a}_s)$. For $r \in R_s$, $s \subset t$, we define $Y_r = I(\theta_r[b,\bar{a}_s])$. For every $s \subset t$, from $N \models \theta_s(b,\bar{a}_s)$ it follows that $B(D) \models \psi_s(Y_{\langle s,1\rangle}, Y_{\langle s.2\rangle}, \cdots,)$. Also, for every $w,v \subset \{\langle s,j\rangle: s \subset t\}$, if $i \in \bigcap_{r \in w} Y_r^c \cap \bigcap_{r \in v} Y_r^c$ then $M_i \models \bigwedge_{r \in w} \theta_r(b(i),\ \bar{a}_r(i)) \wedge \bigwedge_{r \in v} \neg\theta_r(b(i),\ \bar{a}_r(i)t$. So $M_i \models (\exists x) \bigwedge_{r \in w} \theta_r(x,\bar{a}_r(i)) \wedge \bigwedge_{r \in v} \neg\, \theta_r(x,\bar{a}_r(i))$, and therefore $i \in f(w_,^c v)$. Hence $\bigcap_{r \in w} Y_r \cap \bigcap_{r \in v} Y_r^c \subset f(w,v)$.

So the set of formulas mentioned above is satisfiable, and hence the set of formulas from (*) is finitely satisfiable, and, as mentioned above, (by the λ-saturation of $(B(D))$ (*) holds.

Now we shall prove

(**) *There exist subsets of I, A_r for $r \in R$ such that:*

(1) *for every r, $B(D) \models A_r = Y_r$ (the Y_r are from (*) and so $B(D) \models \psi_s [Y_{\langle s,1\rangle}, \cdots,])$.*

(2) *for each $w,\ v \in S_w(R)$, $\bigcap_{r \in w} y_r \cap \bigcap_{r \in v} Y_r^c \subset f(w,v)$.*

As we have mentioned already, $|R| = \mu < \lambda$, and f satisfy condition (2) of Lemma 2.5 (by its definition). Recalling also condition (2) of (*), we see that the hypotheses of Lemma 2.5 are satisfied if we take R instead of μ (as the set of indices) and Y_r instead of A_k (sequence of known sets). So by Lemma 2.5, (**) is true.

Now it is easily seen that for each i, the set $q_i = \{\theta_r(x,\bar{a}_r(i)): r \in R,\ i \in A_r\}$

$\cup\{\neg\theta_r(x,\bar{a}_r(i)) \colon r \in R,\ i \in A_r\}$ is finitely satisfiable. (This is clear from (**) and the definition of f.) If we define b such that $b(i)$ satisfies q_i, then for every r, $I(\theta_r[b,\bar{a}_r]) = A_r$, and so for every s, $B(D) \models \psi_s[I(\theta_{\langle s,1\rangle}[b,\bar{a}_s]),\cdots,]$. Hence $N \models \theta_s(b,\bar{a}_s)$, and so for every $k < \mu$ (by taking $s = \{k\}$) $N \models \phi_k(b,\bar{a}_k)$.

But although q_i is finitely satisfiable, perhaps it is not satisfiable. Since $\mu < \lambda$, D is (ω,μ) regular. As $|R| = \mu$, there are sets X_r, $r \in R$, in D, the intersection of any infinite number of them is void. Let

$$q_i^1 = \{\theta_r(x,\bar{a}_r) \colon r \in R,\ i \in A_r \cap X_r\} \cup \{\neg\theta_r(x,\bar{a}_r) \colon r \in R,\ i \notin A_r,\ i \in X_r\}.$$

Clearly $q_i^1 \subset q_i$, and by the definition of the X_r, each q_i^1 is finite (otherwise i will belong to infinitely many X_r's). So q_i^1 is satisfied (in M) by $b(i)$. Then for every $r \in R$ $I(\theta_r[b,\bar{a}_r]) \cap X_r = A_r \cap X_r$, and so $B(D) \models I(\theta_r[b,\bar{a}_r]) = A_r$. Hence $B(D) \models \psi_s[I(\theta_{\langle s,1\rangle}[b,\bar{a}_s]),\cdots,]$ for each s, and so as before, $N \models \phi_k[b,\bar{a}_k]$ for each k.

Thus we end the proof of Theorem 2.1.

THEOREM 2.7. *If $\mu \leqq |I|$, then for every M, $|L(M)| \leqq \mu$, M^I/D is μ-universal iff D is (ω,μ)-regular and $B(D)$ is μ-universal.*

REMARK. This generalizes a parallel theorem for ultrapower.

PROOF. As the proof can be constructed easily from the proof of Theorem 2.1, we do not repeat it.

REFERENCES

1. M. Benda, *On saturated reduced direct products*, Notices Amer. Math. Soc. **16**, 842.
2. T. Frayne, A. Morel, and D. Scott, *Reduced direct products*, Fund. Math. **51** (1962) 195–248.
3. H. J. Keisler, *Limit ultrapowers*, Trans, Amer. Math. Soc. **107** (1963), 382–408.
4. H. J. Keisler, *Ultraproducts and saturated models*, Indag. Math. **26** (1964), 178–186.
5. H. J. Keisler, *Good ideals in fields of sets*, Ann. of Math. **79** (1964), 338–359.
6. H. J. Keisler, *Ultraproducts which are not saturated*, J. Symbolic Logic **32** (1969), 23–46.
7. S. Kochen, *Ultraproducts in the theory of models*, Ann. of Math. **74** (1961), 221–261.
8. J. Los, *Quelques remarques, théorèmes et problèmes sur les classes définissable d'algèbras*, Mathematical interpretations of formal systems by Th. Skolem et al., pp. 98–115 Amsterdam, 1955.
9. L. Pacholski and C. Ryll-Nardzewski, *On countably compact reduced products* (to appear in Fund. Math.).
10. S. Feferman and R. L. Vaught, *Products of algebraic systems*, Fund. Math. **47** (1959), 55–103.
11. E. Galvin, *Certain reduced products are countably saturated*, Notices Amer. Math. Soc. **14** (1967), 838.

12. S. Shelah, *When every reduced product is saturated*, Notices Amer. Math. Soc. **17** (1970), 453.

13. B. Jonsson and P. Olin, *Almost direct products and saturation*, Compositio Math. **20** (1968), 125–132.

14. L. Pacholski, *On countably compact reduced products III*, Colloq. Math **XXIII** (1971), 5–15.

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